

\mathcal{D} -annihilators, weakly n -angulated categories and derived equivalences

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Abstract

In this paper, we introduce \mathcal{D} -annihilators and weakly n -angulated categories, and give a general construction of derived equivalences between the quotient rings of endomorphism rings (modulo \mathcal{D} -annihilators or proper \mathcal{D} -annihilators) of certain objects involved in a sequence in an additive category or a weakly n -angulated category. This method generalizes the main results in [HX11], [HKX13] and [Che13].

1 Introduction

The theory of derived categories were first developed by Grothendieck and Verdier in early 1960s, and were of great importance in the development of algebraic geometry. Derived equivalences, which are equivalences between derived categories, occur nowadays frequently in many branches of mathematics and physics. Examples include Kazhdan-Lusztig conjecture, Broué's abelian defect group conjecture and mirror symmetry over non-commutative geometry. In most situations, derived equivalences for rings are involved. By Rickard's Morita theory of derived categories [Ric89b], two rings are derived equivalent if and only if there is a suitable "tilting complex". Derived equivalences between rings preserve many significant algebraic and geometric invariants such as Hochschild (co)homology, cyclic homology, center and K -theory, etc. In general, it is very hard to find such a suitable "tilting complex" for two given rings. So, a crucial problem is: how and where can we obtain derived equivalences systematically?

One can start from a known derived equivalence, and get new derived equivalences by forming trivial extension and tensor products [Ric89a, Ric91]. A rather different approach was carried out in [HX11]. The results in [HX11] shows that there are a lot derived equivalencies between the endomorphism algebras of the modules involved in Auslander-Reiten sequences. The key point is the notion of \mathcal{D} -split sequences, which has Auslander-Reiten sequences as important examples. This idea was further developed in [HKX13] and [Che13], considering \mathcal{D} -split triangles and taking certain cohomological approximations into play.

In this paper, we carry out a more general approach: constructing derived equivalences between the endomorphism rings, or its quotient/sub rings from a sequence in an additive category or a weakly n -angulated category (see Definition 5.1 below). The key notions here are \mathcal{D} -annihilators and proper \mathcal{D} -annihilators, which are certain canonical ideals of an additive category defined as follows. Let \mathcal{C} be an additive category, and let \mathcal{D} be an additive full subcategory of \mathcal{C} . For two objects $A, B \in \mathcal{C}$, the left \mathcal{D} -annihilator, denoted by $\mathcal{L}_{\mathcal{D}}(A, B)$, consists of morphisms $f \in \mathcal{C}(A, B)$ such that $\mathcal{C}(\mathcal{D}, f)$ is zero. The left proper \mathcal{D} -annihilator, denoted by $\mathcal{I}_{\mathcal{D}}(A, B)$, consists of those morphisms f in $\mathcal{L}_{\mathcal{D}}(A, B)$ factorizing through an object in \mathcal{D} . Similarly, we have the right \mathcal{D} -annihilator $\mathcal{R}_{\mathcal{D}}(A, B)$ and the right proper \mathcal{D} -annihilator $\mathcal{J}_{\mathcal{D}}(A, B)$.

Our first main result can be stated as follows.

Theorem 1.1. *Let \mathcal{C} be an additive k -category, and let $\mathcal{D} = \text{add}(M)$ for an object M in \mathcal{C} . Suppose that $n \geq 1$ is an integer and that Q^\bullet :*

$$0 \longrightarrow X \xrightarrow{d^0} Q^1 \xrightarrow{d^1} Q^2 \longrightarrow \cdots \longrightarrow Q^n \xrightarrow{d^n} Y \longrightarrow 0$$

is a complex over \mathcal{C} with $Q^i \in \mathcal{D}$ for all $1 \leq i \leq n$ and X in degree zero. Assume that the following conditions are satisfied:

- (1). $H^i(\text{Hom}_{\mathcal{C}}(M, Q^\bullet)) = 0$ for all $i \neq 0$;
- (2). $H^i(\text{Hom}_{\mathcal{C}}(Q^\bullet, M)) = 0$ for all $i \neq -n-1$;
- (3). $H^1(\text{Hom}_{\mathcal{C}}^\bullet(X, Q^\bullet)) = 0 = H^{-n}(\text{Hom}_{\mathcal{C}}^\bullet(Q^\bullet, Y))$.

Then the algebras $\text{End}_{\mathcal{C}/\mathcal{L}_{\mathcal{D}}}(M \oplus X)$ and $\text{End}_{\mathcal{C}/\mathcal{R}_{\mathcal{D}}}(M \oplus Y)$ are derived equivalent.

Let us remark that an \mathcal{D} -split sequence $X \rightarrow M \rightarrow Y$ automatically satisfies the conditions on Theorem 1.1, and $\mathcal{L}_{\mathcal{D}}(X \oplus M, X \oplus M) = 0 = \mathcal{R}_{\mathcal{D}}(M \oplus Y, M \oplus Y)$. In this case, Theorem 1.1 tells us that $\text{End}_{\mathcal{C}}(X \oplus M)$ and $\text{End}_{\mathcal{C}}(M \oplus Y)$ are derived equivalent. Thus Theorem 1.1 generalizes the main result in [HX11].

As a generalization of triangulated category, Geiss et al [GKO13a] introduced n -angulated categories, which occur widely in cluster-tilting theory and are closely related to algebraic geometry and string theory. In this paper, we consider weakly n -angulated categories. Roughly speaking, the relation between a weakly n -angulated category and an n -angulated category is like that between an additive category and an abelian category. In a weakly n -angulated category, we do not have the pushout axiom (Octahedral axiom when $n = 3$) and we do not require every morphism to be embedded into an n -angle (In general, in an additive category, pushout/pullback does not exist, and a morphism does not necessarily have a kernel or cokernel). However, the shifting of an n -angle is still an n -angle, and the direct sum of two n -angles is again an n -angle. For a precise definition of a weakly n -angulated category, please see Definition 5.1 below. Our second main result is as follows.

Theorem 1.2. *Let (\mathcal{C}, Σ) be a weakly n -angulated category, and let $\mathcal{D} := \text{add}(M)$ for some object M in \mathcal{C} . Suppose that*

$$X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_{n-2} \xrightarrow{g} Y \xrightarrow{\eta} \Sigma X$$

is an n -angle in \mathcal{C} such that f and g are left and right \mathcal{D} -approximations, respectively. Then the algebras $\text{End}_{\mathcal{C}/\mathcal{I}_{\mathcal{D}}}(M \oplus X)$ and $\text{End}_{\mathcal{C}/\mathcal{J}_{\mathcal{D}}}(M \oplus Y)$ are derived equivalent.

Comparing with n -angulated categories, an advantage here is that the notion of weakly n -angulated category is very much compatible with the Φ -orbit category construction. Namely, let \mathcal{T} be a weakly n -angulated category, and let F be an n -angulated endo-functor of \mathcal{T} . Suppose that Φ is an admissible subset of \mathbb{Z} . Then the Φ -orbit category $\mathcal{T}^{\Phi, F}$ is again a weakly n -angulated category. The Φ -orbit category of an n -angulated category is not an n -angulated category in general. Moreover, the ideals I and J occurring in [HKX13, Theorem 3.1] and [Che13, Section 3] fit well into our setting here: they are left and right proper \mathcal{D} -annihilators in the Φ -orbit category (see Lemma 5.4), respectively. Thus, Theorem 1.2 has the following corollary.

Corollary 1.3. *Let (\mathcal{T}, Σ) be a weakly n -angulated category with an n -angulated endo-functor F , and let M be an object in \mathcal{T} . Suppose that Φ is an admissible subset of \mathbb{Z} . Let*

$$X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_{n-2} \xrightarrow{g} Y \longrightarrow \Sigma X$$

be an n -angle in \mathcal{T} such that $M_i \in \text{add}(M)$ for all $i = 1, \dots, n-2$, and that f and g are left and right $\text{add}(M)$ -approximations in the orbit category $\mathcal{T}^{F, \Phi}$, respectively. Suppose that $\mathcal{T}(Y, F^i M) = 0 = \mathcal{T}(M, F^i X)$ for all $0 \neq i \in \Phi$. Then the rings $\text{End}_{\mathcal{T}^{F, \Phi}}(M \oplus X)/I$ and $\text{End}_{\mathcal{T}^{F, \Phi}}(M \oplus Y)/J$ are derived equivalent, where the ideals I and J are the above mentioned ideals.

This corollary generalizes the main results of the papers [HKX13] and [Che13], where the endofunctor F was assumed to be an auto-equivalence. Moreover, unlike [HKX13] and [Che13], the approach in this paper avoids the technique calculation of morphisms in Φ -Yoneda algebras.

This paper is organized as follows. In Section 2, we make some preparations, including the Φ -orbit construction. The \mathcal{D} -annihilators and proper \mathcal{D} -annihilators are introduced in Section 3. The following two sections are devoted to constructing derived equivalences in an additive category and a weakly n -angulated category, respectively. In particular, the main results Theorem 1.1 and Theorem 1.2 will be proved.

2 Preliminary

In this section, we shall recall basic definitions and facts which are needed in our proofs.

2.1 Conventions

Throughout this paper, k is a fixed commutative ring with identity. Let \mathcal{C} be an additive k -category, that is, \mathcal{C} is an additive category in which the morphism set of two objects X and Y in \mathcal{C} , denoted by $\mathcal{C}(X, Y)$, is a k -module, and the composition of morphisms in \mathcal{C} is k -bilinear. For an object X in \mathcal{C} , the endomorphism algebra $\mathcal{C}(X, X)$ is denoted by $\text{End}_{\mathcal{C}}(X)$. We write $\text{add}_{\mathcal{C}}(X)$ for the full subcategory of \mathcal{C} consisting of all direct summands of finite direct sums of copies of X . If there is no confusion, we just write $\text{add}(X)$ for $\text{add}_{\mathcal{C}}(X)$. An object X in \mathcal{C} is called an *additive generator* for \mathcal{C} if $\mathcal{C} = \text{add}(X)$. For two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , we write fg for their composite. But for two functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ of categories, we write GF for their composite instead of FG .

All categories in this paper are additive k -categories, and all functors are additive k -functors. Let $\eta : F \rightarrow G$ be a natural transformation between two functors from \mathcal{C} to \mathcal{D} . For an object $X \in \mathcal{C}$, we denote by η_X the morphism from $F(X)$ to $G(X)$ induced by η . If $H : \mathcal{A} \rightarrow \mathcal{C}$ is another functor, then η gives rise to a natural transformation $\eta_H : FH \rightarrow GH$. If H is a functor from \mathcal{D} to \mathcal{E} , then we have a natural transformation $H(\eta) : HF \rightarrow HG$.

Let \mathcal{C} be a category. A functor F from \mathcal{C} to itself is called an *endo-functor* of \mathcal{C} . If there is another endo-functor G of \mathcal{C} such that $FG = GF = \text{id}_{\mathcal{C}}$, where $\text{id}_{\mathcal{C}}$ is the identity functor of \mathcal{C} , then F is called an *automorphism* of \mathcal{C} . F is called an *auto-equivalence* provided that there is another endo-functor G of \mathcal{C} such that both FG and GF are naturally isomorphic to $\text{id}_{\mathcal{C}}$.

2.2 Complexes and derived equivalences

Let \mathcal{C} be an additive k -category. A complex X^\bullet over \mathcal{C} is a sequence of morphisms $\cdots \rightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \cdots$ between objects in \mathcal{C} such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$. The category of complexes over \mathcal{C} with morphisms being chain maps is denoted by $\mathcal{C}(\mathcal{C})$. The homotopy category of complexes over \mathcal{C} is denoted by $\mathcal{K}(\mathcal{C})$. If \mathcal{C} is an abelian category, then the derived category of complexes over \mathcal{C} is denoted by $\mathcal{D}(\mathcal{C})$. We write $\mathcal{C}^b(\mathcal{C})$, $\mathcal{K}^b(\mathcal{C})$ and $\mathcal{D}^b(\mathcal{C})$ respectively for the full subcategories of $\mathcal{C}(\mathcal{C})$, $\mathcal{K}(\mathcal{C})$ and $\mathcal{D}(\mathcal{C})$ consisting of bounded complexes.

It is well known that the categories $\mathcal{K}(\mathcal{C})$ and $\mathcal{D}(\mathcal{C})$ are triangulated categories with $\mathcal{K}^b(\mathcal{C})$ and $\mathcal{D}^b(\mathcal{C})$ being their triangulated full subcategories, respectively. For basic results on triangulated categories, we refer to Happel's book [Hap88]. However, the shifting functor in a triangulated category is written as Σ in this paper.

For two complexes X^\bullet and Y^\bullet over \mathcal{C} , we write $\text{Hom}_{\mathcal{C}}^\bullet(X^\bullet, Y^\bullet)$ for the total complex of the double complex with the (i, j) -term (X^{-j}, Y^i) .

Let Λ be a ring with identity. The category of left Λ -modules, denoted by $\Lambda\text{-Mod}$, is an abelian category. The full subcategory of $\Lambda\text{-Mod}$ consisting of finitely generated projective Λ -modules is denoted by $\Lambda\text{-proj}$. Following [Ric89b], two rings Λ and Γ are said to be *derived equivalent* provided that the derived categories $\mathcal{D}^b(\Lambda\text{-Mod})$ and $\mathcal{D}^b(\Gamma\text{-Mod})$ of bounded complexes are equivalent as triangulated categories. Due to the work of Rickard [Ric89b] (see also [Kel94]), two rings Λ and Γ are derived equivalent if and only if there is a bounded complex T^\bullet of finitely generated projective Λ -modules satisfying the following two conditions,

- (a). T^\bullet is self-orthogonal, that is, $\mathcal{K}^b(\Lambda)(T^\bullet, \Sigma^i T^\bullet) = 0$ for all $i \neq 0$;
- (b). $\text{add}(T^\bullet)$ generates $\mathcal{K}^b(\Lambda\text{-proj})$ as a triangulated category,

such that $\text{End}_{\mathcal{K}^b(\Lambda\text{-proj})}(T^\bullet)$ is isomorphic to Γ as rings. A complex T^\bullet in $\mathcal{K}^b(\Lambda\text{-proj})$ satisfying the above two conditions is called a *tilting complex* over Λ .

2.3 Admissible sets and Φ -orbit categories

Let us recall from [HX13] and [HKX13] the definition of admissible subsets. A subset Φ of \mathbb{Z} containing 0 is called an *admissible subset* provided that the following property holds:

If $i + j + k \in \Phi$ for three elements i, j, k in Φ , then $i + j \in \Phi$ if and only if $j + k \in \Phi$

Typical examples of admissible subsets of \mathbb{Z} include $n\mathbb{Z}$ and $\{0, 1, \dots, n\}$. Suppose that Φ is an admissible subset of \mathbb{Z} . Then $-\Phi := \{-i | i \in \Phi\}$, $\Phi^{\geq 0} := \{i \in \Phi | i \geq 0\}$ and $\Phi^{\leq 0} := \{i \in \Phi | i \leq 0\}$ are all admissible. Let m be an integer. The set $m\Phi := \{mi | i \in \Phi\}$ is admissible. Moreover, if $m \geq 3$, then the set $\Phi^m := \{i^m | i \in \Phi\}$ is admissible. Nevertheless, not all subsets of \mathbb{Z} containing zero are admissible. For instance, the set $\{0, 1, 2, 4\}$ is not admissible.

Now let \mathcal{T} be an additive k -category, and let F be an endo-functor of \mathcal{T} . If F is not an equivalence, we set $F^i = 0$ for all $i < 0$. If F is an equivalence, we fix a quasi-inverse F^{-1} of F , and set $F^i := (F^{-1})^{-i}$ for $i < 0$. The functor F^0 is defined to be the identity functor on \mathcal{T} . We can define a category $\mathcal{T}^{F, \Phi}$ of \mathcal{T} as follows. The objects in $\mathcal{T}^{F, \Phi}$ are the same as \mathcal{T} , and the morphism space $\mathcal{T}^{F, \Phi}(X, Y)$ for two objects X, Y is defined to be

$$\bigoplus_{i \in \Phi} \mathcal{T}(X, F^i Y).$$

In [HKX13], for each pair of integers u and v , a natural transformation $\chi(u, v)$ from $F^u F^v$ to F^{u+v} is defined, and it is proved that the composition

$$\mathcal{T}(X, F^u Y) \times \mathcal{T}(Y, F^v Z) \longrightarrow \mathcal{T}(X, F^{u+v} Z),$$

sending (f_u, g_v) to $f_u * g_v := f_u F^u(g_v) \chi(u, v)_Z$, is associative. We refer to [HKX13, 2.3] for the details of the natural transformations $\chi(u, v) : F^u F^v \longrightarrow F^{u+v}$. As a result, for morphisms $f = (f_i)_{i \in \Phi} \in \mathcal{T}^{F, \Phi}(X, Y)$ and $g = (g_i)_{i \in \Phi} \in \mathcal{T}^{F, \Phi}(Y, Z)$, the composition

$$(f, g) \mapsto fg := \left(\sum_{\substack{u, v \in \Phi \\ u+v=i}} f_u * g_v \right)_{i \in \Phi}$$

is associative. Thus $\mathcal{T}^{F, \Phi}$ is indeed an additive k -category, and is called the *Φ -orbit category* of \mathcal{T} under the functor F . The endomorphism algebra of an object X in $\mathcal{T}^{F, \Phi}$ is denoted by $E_{\mathcal{T}}^{F, \Phi}(X)$, and is called the *Φ -Yoneda algebra* of X with respect to F .

For each $X, Y \in \mathcal{T}$, the morphism space $\mathcal{T}^{F, \Phi}(X, Y) = \bigoplus_{i \in \Phi} \mathcal{T}(X, F^i Y)$ is Φ -graded. Every morphism $\alpha \in \mathcal{T}(X, F^i Y)$ can be viewed as a homogeneous morphism in $\mathcal{T}^{F, \Phi}(X, Y)$ of degree i .

Suppose that F is an auto-equivalence of \mathcal{T} . If both i and $-i$ are in the admissible subset Φ , then X and $F^i X$ are isomorphic in the Φ -orbit category $\mathcal{T}^{F,\Phi}$. Actually, let $f := \chi(-i, i)_X^{-1} : X \rightarrow F^{-i}(F^i X)$ and $g := 1_{F^i X} : F^i X \rightarrow F^i X$. Considering f as a homogeneous morphism in $\mathcal{T}^{F,\Phi}$ of degree $-i$, and g as a homogeneous morphism in $\mathcal{T}^{F,\Phi}$ of degree i , we have $f * g = 1_X$ and $g * f = 1_{F^i X}$.

2.4 Approximations and cohomological approximations

Now we recall some definitions from [AS80].

Let \mathcal{C} be a category, and let \mathcal{D} be a full subcategory of \mathcal{C} , and X an object in \mathcal{C} . A morphism $f : D \rightarrow X$ in \mathcal{C} is called a *right \mathcal{D} -approximation* of X if $D \in \mathcal{D}$ and the induced map $\text{Hom}_{\mathcal{C}}(D', f) : \text{Hom}_{\mathcal{C}}(D', D) \rightarrow \text{Hom}_{\mathcal{C}}(D', X)$ is surjective for every object $D' \in \mathcal{D}$. Dually, there is the notion of a *left \mathcal{D} -approximation*. The subcategory \mathcal{D} is called *contravariantly* (respectively, *covariantly*) *finite* in \mathcal{C} if every object in \mathcal{C} has a right (respectively, left) \mathcal{D} -approximation. The subcategory \mathcal{D} is called *functorially finite* in \mathcal{C} if \mathcal{D} is both contravariantly and covariantly finite in \mathcal{C} .

Cohomological approximations are introduced in [HKX13]. Let \mathcal{T} be an additive k -category, and let F be a functor from \mathcal{T} to itself. Suppose that Φ is a non-empty subset of \mathbb{Z} , and that \mathcal{D} is a full additive subcategory of \mathcal{T} . A morphism $f : X \rightarrow D^X$ in \mathcal{T} with $D^X \in \mathcal{D}$ is called a *left (\mathcal{D}, F, Φ) -approximation* if every morphism $X \rightarrow F^i D$, where $D \in \mathcal{D}$ and $i \in \Phi$, factorizes through f . In case that Φ is an admissible subset, we have the Φ -orbit category $\mathcal{T}^{F,\Phi}$, and that f is a left (\mathcal{D}, F, Φ) -approximation is equivalent to saying that $\mathcal{T}^{F,\Phi}(D^X, \mathcal{D}) \rightarrow \mathcal{T}^{F,\Phi}(X, \mathcal{D})$ is surjective, i.e., the morphism f , as a homogeneous morphism of degree zero, is a left \mathcal{D} -approximation in the orbit category $\mathcal{T}^{F,\Phi}$.

In [HKX13], a right (\mathcal{D}, F, Φ) -approximation is defined to be a morphism $g : D_Y \rightarrow Y$ in \mathcal{T} with $D_Y \in \mathcal{D}$ such that every morphism from $F^i D$ to Y with $i \in \Phi$ and $D \in \mathcal{D}$ factorizes through g . Unfortunately, this does not fit the Φ -orbit category well: g is NOT a right \mathcal{D} -approximation in the orbit category $\mathcal{T}^{F,\Phi}$ in general. However, when F is an auto-equivalence with a quasi-inverse F^{-1} , a right $(\mathcal{D}, F, -\Phi)$ -approximation is still a right \mathcal{D} -approximation in $\mathcal{T}^{F,\Phi}$. Here we re-define a right (\mathcal{D}, F, Φ) -approximation as follows.

A morphism $g : D_Y \rightarrow Y$ in \mathcal{T} with $D_Y \in \mathcal{D}$ is called a right (\mathcal{D}, F, Φ) -approximation if every morphism from D to $F^i Y$ with $i \in \Phi$ and $D \in \mathcal{D}$ factorizes through g .

Suppose that Φ is an admissible subset of \mathbb{Z} . Thus, a morphism $g : D_Y \rightarrow Y$ in \mathcal{T} is a right (\mathcal{T}, F, Φ) -approximation if and only if g is a right \mathcal{D} -approximation in the Φ -orbit category $\mathcal{T}^{F,\Phi}$, no matter F is an equivalence or not.

3 \mathcal{D} -annihilators and proper \mathcal{D} -annihilators

Let \mathcal{C} be an additive k -category. By an *ideal* \mathcal{I} on \mathcal{C} we mean k -submodules $\mathcal{I}(A, B) \subseteq \mathcal{C}(A, B)$ for all A and B in \mathcal{C} , such that the composite $\alpha\beta$ belongs to \mathcal{I} provided either α or β is in \mathcal{I} . We denote $\mathcal{I}(A, A)$ simply by $\mathcal{I}(A)$. Let \mathcal{D} be an additive full subcategory of \mathcal{C} . Then we get the following three ideals of \mathcal{C} by defining, for each pair of objects A, B in \mathcal{C} ,

$$\mathcal{R}_{\mathcal{D}}(A, B) := \{f \in \mathcal{C}(A, B) \mid \mathcal{C}(f, \mathcal{D}) = 0\}$$

$$\mathcal{L}_{\mathcal{D}}(A, B) := \{f \in \mathcal{C}(A, B) \mid \mathcal{C}(\mathcal{D}, f) = 0\}.$$

$$\mathcal{F}_{\mathcal{D}}(A, B) := \{f \in \mathcal{C}(A, B) \mid f \text{ factorizes through an object in } \mathcal{D}\}$$

Moreover, we set

$$\mathcal{I}_{\mathcal{D}} := \mathcal{L}_{\mathcal{D}} \cap \mathcal{F}_{\mathcal{D}} \text{ and } \mathcal{J}_{\mathcal{D}} := \mathcal{R}_{\mathcal{D}} \cap \mathcal{F}_{\mathcal{D}}.$$

The ideal $\mathcal{L}_{\mathcal{D}}$ is called the *left \mathcal{D} -annihilator* and the ideal $\mathcal{I}_{\mathcal{D}}$ is called the *left proper \mathcal{D} -annihilator*. Similarly, the ideal $\mathcal{R}_{\mathcal{D}}$ is called the *right \mathcal{D} -annihilator* and the ideal $\mathcal{J}_{\mathcal{D}}$ is called the *right proper \mathcal{D} -annihilator*.

Lemma 3.1. *Keeping the notations above, we have the following:*

(1). *If $A \in \mathcal{C}$ admits a right \mathcal{D} -approximation $f_A : D_A \rightarrow A$, then*

$$\mathcal{R}_{\mathcal{D}}(A, B) = \{g \in \mathcal{C}(A, B) \mid f_A g = 0\};$$

(2) *If $B \in \mathcal{C}$ admits a left \mathcal{D} -approximation $f^B : B \rightarrow D^B$, then*

$$\mathcal{L}_{\mathcal{D}}(A, B) = \{g \in \mathcal{C}(A, B) \mid g f^B = 0\};$$

(3). *If $A \in \mathcal{D}$, then $\mathcal{R}_{\mathcal{D}}(A, B) = 0$ and $\mathcal{L}_{\mathcal{D}}(A, B) = \mathcal{I}_{\mathcal{D}}(A, B)$;*

(4). *If $B \in \mathcal{D}$, then $\mathcal{L}_{\mathcal{D}}(A, B) = 0$ and $\mathcal{R}_{\mathcal{D}}(A, B) = \mathcal{J}_{\mathcal{D}}(A, B)$.*

Proof. (1). Let $g : A \rightarrow B$ be in $\mathcal{R}_{\mathcal{D}}(A, B)$. Then $\mathcal{C}(\mathcal{D}, g) = 0$, and particularly $\mathcal{C}(D_A, g) = 0$. Consequently $f_A g = 0$. Conversely, let g be in $\mathcal{C}(A, B)$ such that $f_A g = 0$. It follows that $0 = \mathcal{C}(D', f_A g) = \mathcal{C}(D', f_A) \mathcal{C}(D', g)$ for all $D' \in \mathcal{D}$. Moreover, since f_A is a right \mathcal{D} -approximation, the morphism $\mathcal{C}(D', f_A)$ is surjective. Hence $\mathcal{C}(D', g) = 0$ for all $D' \in \mathcal{D}$, that is, $g \in \mathcal{R}_{\mathcal{D}}(A, B)$.

The proof of (2) is dual to that of (1).

(3). Suppose that $A \in \mathcal{D}$. The identity map $1_A : A \rightarrow A$ is a right \mathcal{D} -approximation. It follows from (1) that $\mathcal{R}_{\mathcal{D}}(A, B) = 0$. Clearly, all the morphisms in $\mathcal{C}(A, B)$ factorizes through the object A , which is in \mathcal{D} . This means that $\mathcal{F}_{\mathcal{D}}(A, B) = \mathcal{C}(A, B)$. Thus $\mathcal{I}_{\mathcal{D}}(A, B) = \mathcal{L}_{\mathcal{D}}(A, B) \cap \mathcal{F}_{\mathcal{D}}(A, B) = \mathcal{L}_{\mathcal{D}}(A, B)$. Similarly, we can prove (4). \square

Example. Suppose that \mathcal{C} is the the module category of finitely generated left modules over an artin algebra A .

(a). Set $\mathcal{D} := \text{add}(M)$ for some A -module M . (i). If M is a generator for A , then $\mathcal{R}_{\mathcal{D}} = 0$; (ii). If M is a co-generator for A , then $\mathcal{L}_{\mathcal{D}} = 0$; (iii). If M is a generator-cogenerator, then $\mathcal{L}_{\mathcal{D}} = 0 = \mathcal{R}_{\mathcal{D}}$.

(b). Let $\mathcal{D} := \text{add}(A/\text{rad}(A))$. Then $\mathcal{R}_{\mathcal{D}}(X, Y)$ consists of morphisms f factorizing through the canonical map $X \rightarrow X/\text{soc}(X)$, and $\mathcal{L}_{\mathcal{D}}(X, Y)$ consists of morphisms g with image contained in $\text{rad}(Y)$. If P is a projective A -module, then $\mathcal{L}_{\mathcal{D}}(P)$ is just the Jacobson radical $\text{rad}(\text{End}_A(P))$ of $\text{End}_A(P)$. Similarly, if I is an injective A -module, then $\mathcal{R}_{\mathcal{D}}(I)$ is just the Jacobson radical of $\text{End}_A(I)$.

Further, we have the following useful lemma.

Lemma 3.2. *Let \mathcal{C} be an additive k -category, and let $\mathcal{D} := \text{add}(M)$ for some object M in \mathcal{C} . Suppose that P^\bullet is a bounded complex*

$$0 \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \dots \longrightarrow P^{n-1} \xrightarrow{d^{n-1}} P^n \longrightarrow 0$$

over \mathcal{C} such that $P^i \in \mathcal{D}$ for all $i > 0$. Assume that the following two conditions are satisfied:

(1). $H^i(\text{Hom}_{\mathcal{C}}^\bullet(M, P^\bullet)) = 0$ for all $i \neq 0, n$;

(2). $H^i(\text{Hom}_{\mathcal{C}}^\bullet(P^\bullet, M)) = 0$ for all $i \neq -n$.

Then P^\bullet is self-orthogonal as a complex both in $\mathcal{K}^b(\mathcal{C}/\mathcal{L}_{\mathcal{D}})$ and in $\mathcal{K}^b(\mathcal{C}/\mathcal{I}_{\mathcal{D}})$.

Proof. For simplicity, we denote by $\overline{\mathcal{C}}$ the category $\mathcal{C}/\mathcal{I}_{\mathcal{D}}$, and denote by $\overline{\overline{\mathcal{C}}}$ the category $\mathcal{C}/\mathcal{L}_{\mathcal{D}}$.

If $n = 0$, then problem is trivial. Now we assume that $n > 0$.

It follows from our assumption (2) that $H^0(\text{Hom}_{\mathcal{C}}^{\bullet}(P^{\bullet}, M)) = 0$, and consequently the map $\mathcal{C}(d^0, M) : \mathcal{C}(P^1, M) \rightarrow \mathcal{C}(P^0, M)$ is surjective. Thus, the morphism d^0 is a left \mathcal{D} -approximation. By Lemma 3.1 (2), one has $\mathcal{L}_{\mathcal{D}}(M, P^0) = \{f \in \mathcal{C}(M, P^0) | f d^0 = 0\} = \text{Ker } \mathcal{C}(M, d^0)$. Moreover, it follows from Lemma 3.1 (3) that $\mathcal{L}_{\mathcal{D}}(M, P^0) = \mathcal{I}_{\mathcal{D}}(M, P^0)$. Hence the canonical functors $\mathcal{C} \rightarrow \overline{\mathcal{C}} \rightarrow \overline{\overline{\mathcal{C}}}$ induced isomorphisms

$$\mathcal{C}(M, P^0) / \text{Ker } \mathcal{C}(M, d^0) \xrightarrow{\pi^0} \overline{\mathcal{C}}(M, P^0) \xrightarrow{p^0} \overline{\overline{\mathcal{C}}}(M, P^0).$$

Note that for each $i > 0$, by Lemma 3.1 (4), we have $\mathcal{L}_{\mathcal{D}}(M, P^i) = 0 = \mathcal{I}_{\mathcal{D}}(M, P^i)$ since $P^i \in \mathcal{D}$. Thus, for each $i > 0$, the canonical functors $\mathcal{C} \rightarrow \overline{\mathcal{C}} \rightarrow \overline{\overline{\mathcal{C}}}$ also induce isomorphisms

$$\mathcal{C}(M, P^i) \xrightarrow{\pi^i} \overline{\mathcal{C}}(M, P^i) \xrightarrow{p^i} \overline{\overline{\mathcal{C}}}(M, P^i).$$

In this way, we see that the complexes $\text{Hom}_{\mathcal{C}}^{\bullet}(M, P^{\bullet})$ and $\text{Hom}_{\overline{\mathcal{C}}}^{\bullet}(M, P^{\bullet})$ are both isomorphic to the complex

$$0 \rightarrow \mathcal{C}(M, P^0) / \text{Ker } \mathcal{C}(M, d^0) \rightarrow \mathcal{C}(M, P^1) \rightarrow \dots \rightarrow \mathcal{C}(M, P^n) \rightarrow 0.$$

By assumption (1), the above complex has zero homology for all degrees not equal to n . Hence $H^i(\text{Hom}_{\overline{\mathcal{C}}}^{\bullet}(M, P^{\bullet})) = 0 = H^i(\text{Hom}_{\mathcal{C}}^{\bullet}(M, P^{\bullet}))$ for all $i \neq n$. By Lemma 3.1 (4), we have $\mathcal{L}_{\mathcal{D}}(P^i, M) = 0$ for all i , and therefore $\mathcal{I}_{\mathcal{D}}(P^i, M) = 0$ for all i . Hence the complexes $\text{Hom}_{\mathcal{C}}^{\bullet}(M, P^{\bullet})$, $\text{Hom}_{\overline{\mathcal{C}}}^{\bullet}(M, P^{\bullet})$ and $\text{Hom}_{\overline{\overline{\mathcal{C}}}}^{\bullet}(M, P^{\bullet})$ are all isomorphic. Hence $H^i(\text{Hom}_{\mathcal{C}}^{\bullet}(M, P^{\bullet})) = 0 = H^i(\text{Hom}_{\overline{\mathcal{C}}}^{\bullet}(M, P^{\bullet}))$ for all $i \neq -n$. The lemma then follows from the result [HK03, Lemma 2.1]. \square

The following lemma is a dual of the above lemma.

Lemma 3.3. *Let \mathcal{C} be an additive k -category, and let $\mathcal{D} := \text{add}(M)$ for some object M in \mathcal{C} . Suppose that P^{\bullet} is a bounded complex*

$$0 \rightarrow P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \rightarrow \dots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \rightarrow 0$$

over \mathcal{C} such that $P^i \in \mathcal{D}$ for all $i < 0$. Assume that the following two conditions are satisfied:

- (1). $H^i(\text{Hom}_{\mathcal{C}}^{\bullet}(M, P^{\bullet})) = 0$ for all $i \neq -n$;
- (2). $H^i(\text{Hom}_{\mathcal{C}}^{\bullet}(P^{\bullet}, M)) = 0$ for all $i \neq 0, n$.

Then P^{\bullet} is self-orthogonal as a complex both in $\mathcal{K}^b(\mathcal{C}/\mathcal{R}_{\mathcal{D}})$ and in $\mathcal{K}^b(\mathcal{C}/\mathcal{I}_{\mathcal{D}})$.

4 Derived equivalences and \mathcal{D} -annihilators

In this section, we consider sequences in an additive category, and give a general method to construct derived equivalences between the endomorphism rings of certain objects involved in the sequences. This idea started from [HX11], where \mathcal{D} -split sequences were introduced.

Let us recall the definition from [HX11]. Let \mathcal{C} be an additive category, and let \mathcal{D} be a full subcategory of \mathcal{C} . A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

in \mathcal{C} is called a \mathcal{D} -split sequence if

- (1). $M \in \mathcal{D}$;
- (2). f is a left \mathcal{D} -approximation, and g is a right \mathcal{D} -approximation;

(3). f is a kernel of g , and g is a cokernel of f .

It was proved in [HX11] that the endomorphism rings $\text{End}_{\mathcal{C}}(X \oplus M)$ and $\text{End}_{\mathcal{C}}(M \oplus Y)$ are derived equivalent (via a tilting module) if there is an $\text{add}(M)$ -split sequence $X \rightarrow M' \rightarrow Y$ in \mathcal{C} . Auslander-Reiten sequences in the module category of an artin algebra are typical examples of \mathcal{D} -split sequences. Thus, one can get a lot of derived equivalences from Auslander-Reiten sequences.

In this section, we consider longer sequences in an additive category. Let \mathcal{C} be an additive k -category, and $\mathcal{D} := \text{add}(M)$ for some object $M \in \mathcal{C}$. Suppose that

$$0 \rightarrow X \xrightarrow{f} Q^1 \rightarrow \dots \rightarrow Q^n \xrightarrow{g} Y \rightarrow 0$$

is a complex in \mathcal{C} with X in degree zero and $Q^i \in \mathcal{D}$ for all $1 \leq i \leq n$. We denote this complex by Q^\bullet , and assume that the following conditions are satisfied.

- (1). $H^i(\text{Hom}_{\mathcal{C}}(M, Q^\bullet)) = 0$ for all $i \neq 0$;
- (2). $H^i(\text{Hom}_{\mathcal{C}}(Q^\bullet, M)) = 0$ for all $i \neq -n-1$;
- (3). $H^1(\text{Hom}_{\mathcal{C}}^\bullet(X, Q^\bullet)) = 0 = H^{-n}(\text{Hom}_{\mathcal{C}}^\bullet(Q^\bullet, Y))$.

Our first main result Theorem 1.1 says that $\text{End}_{\mathcal{C}/\mathcal{L}_{\mathcal{D}}}(X \oplus M)$ and $\text{End}_{\mathcal{C}/\mathcal{R}_{\mathcal{D}}}(M \oplus Y)$ are derived equivalent. Let us explain a little on the above conditions. If Q^\bullet is luckily to be an \mathcal{D} -split sequence, then the conditions clearly hold, the left \mathcal{D} -annihilator $\mathcal{L}_{\mathcal{D}}(X \oplus M)$ and the right \mathcal{D} -annihilator $\mathcal{R}_{\mathcal{D}}(M \oplus Y)$ are zero, and we get a derived equivalence between $\text{End}_{\mathcal{C}}(X \oplus M)$ and $\text{End}_{\mathcal{C}}(M \oplus Y)$. The condition (1) implies that g is a right \mathcal{D} -approximation, and the condition (2) implies that f is a left \mathcal{D} -approximation.

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let P^\bullet be the complex

$$0 \rightarrow X \xrightarrow{d^0} Q^1 \rightarrow \dots \rightarrow Q^n \oplus M \xrightarrow{\tilde{d}^n} Y \oplus M \rightarrow 0,$$

where $\tilde{d}^n = \begin{bmatrix} d^n & 0 \\ 0 & 1_M \end{bmatrix}$ is the direct sum of d^n and 1_M . Note that P^\bullet is isomorphic to Q^\bullet in $\mathcal{K}^b(\mathcal{C})$. Hence we have the following:

- a). $H^i(\text{Hom}_{\mathcal{C}}(M, P^\bullet)) = 0$ for all $i \neq 0$;
- b). $H^i(\text{Hom}_{\mathcal{C}}(P^\bullet, M)) = 0$ for all $i \neq -n-1$;
- c). $H^1(\text{Hom}_{\mathcal{C}}^\bullet(X, P^\bullet)) = 0 = H^{-n}(\text{Hom}_{\mathcal{C}}^\bullet(P^\bullet, Y \oplus M))$.

Let T^\bullet be the truncated complex of P^\bullet :

$$0 \rightarrow X \xrightarrow{d^0} Q^1 \rightarrow \dots \rightarrow Q^n \oplus M \rightarrow 0.$$

Then it follows immediately that $H^i(\text{Hom}_{\mathcal{C}}(M, T^\bullet)) = 0$ for all $i \neq 0, n$, and that $H^i(\text{Hom}_{\mathcal{C}}(T^\bullet, M)) = 0$ for all $i \neq -n$. Thus, it follows from Lemma 3.2 that T^\bullet is self-orthogonal in $\mathcal{K}^b(\mathcal{C}/\mathcal{L}_{\mathcal{D}})$. Using the full embedding

$$\text{Hom}_{\mathcal{C}/\mathcal{L}_{\mathcal{D}}}^\bullet(M \oplus X, -) : \mathcal{K}^b(\text{add}_{\mathcal{C}/\mathcal{L}_{\mathcal{D}}}(M \oplus X)) \rightarrow \mathcal{K}^b(\text{End}_{\mathcal{C}/\mathcal{L}_{\mathcal{D}}}(M \oplus X)\text{-proj}),$$

we see that $\tilde{T}^\bullet := \text{Hom}_{\mathcal{C}/\mathcal{L}_{\mathcal{D}}}^\bullet(M \oplus X, T^\bullet)$ is self-orthogonal in $\mathcal{K}^b(\text{End}_{\mathcal{C}/\mathcal{L}_{\mathcal{D}}}(M \oplus X)\text{-proj})$. Moreover, $\text{add}(\tilde{T}^\bullet)$ clearly generates $\mathcal{K}^b(\text{End}_{\mathcal{C}/\mathcal{L}_{\mathcal{D}}}(M \oplus X)\text{-proj})$ as a triangulated category. Hence \tilde{T}^\bullet is a tilting complex over $\text{End}_{\mathcal{C}/\mathcal{L}_{\mathcal{D}}}(M \oplus X)$ with endomorphism algebra isomorphic to $\text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{L}_{\mathcal{D}})}(\tilde{T}^\bullet)$.

By [Ric89b, Theorem 6.4], it remains to prove that $\text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{L}_{\mathcal{D}})}(\tilde{T}^\bullet)$ and $\text{End}_{\mathcal{C}/\mathcal{R}_{\mathcal{D}}}(Y \oplus M)$ are isomorphic.

Firstly, we show that there is a surjective ring homomorphism

$$\theta : \text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet) \longrightarrow \text{End}_{\mathcal{C}/\mathcal{R}_D}(Y \oplus M).$$

For each chain map $f^\bullet : T^\bullet \longrightarrow T^\bullet$ in $\mathcal{C}^b(\mathcal{C})$, it follows from the fact c) above that there is a morphism $g \in \text{End}_{\mathcal{C}}(Y \oplus M)$ such that the following diagram is commutative

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X & \xrightarrow{d^0} & Q^1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & Q^n \oplus M & \xrightarrow{\tilde{d}^n} & Y \oplus M & \longrightarrow & 0 \\ & & \downarrow f^0 & & \downarrow f^1 & & & & \downarrow f^n & & \downarrow g & & \\ 0 & \longrightarrow & X & \xrightarrow{d^0} & Q^1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & Q^n \oplus M & \xrightarrow{\tilde{d}^n} & Y \oplus M & \longrightarrow & 0. \end{array}$$

Moreover, if g' is another morphism in $\text{End}_{\mathcal{C}}(Y \oplus M)$ such that $\tilde{d}^n g' = f^n \tilde{d}^n$, then $\tilde{d}^n(g - g') = 0$. Fact a) above implies that \tilde{d}^n is a right add(M)-approximation of $Y \oplus M$. Thus, by Lemma 3.1 (1), we see that $g - g'$ belongs to $\mathcal{R}_D(Y \oplus M, Y \oplus M)$. We denote by \bar{g} the corresponding morphism of g in $\mathcal{C}/\mathcal{R}_D$. Defining $\theta(f^\bullet) := \bar{g}$ gives rise to a ring homomorphism θ from $\text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet)$ to $\text{End}_{\mathcal{C}/\mathcal{R}_D}(Y \oplus M)$. We claim that θ is surjective. Actually, for each $g \in \text{End}_{\mathcal{C}}(Y \oplus M)$, it follows from the fact a) above that there are morphisms $f^i, i = 1, \dots, n$ such that $f^n \tilde{d}^n = \tilde{d}^n g$ and $f^k d^k = d^k f^{k+1}$ for all $k = 1, \dots, n-1$. The fact c) then allows us to get a morphism $f^0 : X \rightarrow X$ such that $f^0 d^0 = d^0 f^1$. Thus we get a chain map f^\bullet in $\text{End}_{\mathcal{C}}(T^\bullet)$ such that $\theta(f^\bullet) = g$.

Secondly, we claim that there is a surjective ring homomorphism

$$\varphi : \text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet) \longrightarrow \text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{L}_D)}(T^\bullet).$$

Actually, we can define φ to be the composite of the ring homomorphism from $\text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet)$ to $\text{End}_{\mathcal{C}^b(\mathcal{C}/\mathcal{L}_D)}(T^\bullet)$, induced by the canonical functor $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{L}_D$, and the canonical surjective ring homomorphism from $\text{End}_{\mathcal{C}^b(\mathcal{C}/\mathcal{L}_D)}(T^\bullet)$ to $\text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{L}_D)}(T^\bullet)$. Let $f^i : T^i \rightarrow T^i, i = 0, 1, \dots, n$ be morphisms in \mathcal{C} such that f^\bullet is a chain map in $\mathcal{C}^b(\mathcal{C}/\mathcal{L}_D)$. Then $f^i d_T^i - d_T^i f^{i+1} : T^i \rightarrow T^{i+1}$ is in \mathcal{L}_D for all $i = 0, 1, \dots, n-1$. Since $T^i \in \mathcal{D}$ for all $i > 0$, by Lemma 3.1 (4), we get $f^i d_T^i - d_T^i f^{i+1} = 0$ for all $i = 0, 1, \dots, n-1$. Hence f^\bullet is a chain map in $\mathcal{C}^b(\mathcal{C})$, and the canonical map from $\text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet)$ to $\text{End}_{\mathcal{C}^b(\mathcal{C}/\mathcal{L}_D)}(T^\bullet)$ is surjective. Consequently φ is a surjective ring homomorphism.

Finally, we show that θ and φ have the same kernel, which would result in an isomorphism between $\text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{L}_D)}(T^\bullet)$ and $\text{End}_{\mathcal{C}/\mathcal{R}_D}(Y \oplus M)$. By definition, a chain complex $f^\bullet : T^\bullet \longrightarrow T^\bullet$ in $\mathcal{C}^b(\mathcal{C})$ is in $\text{Ker } \varphi$ if and only if there exist $h^i : T^i \rightarrow T^{i-1}, i = 1, \dots, n$ in \mathcal{C} such that $f^n - h^n d_T^{n-1}, f^0 - d_T^0 h^1$ and $f^i - h^i d_T^{i-1} - d_T^i h^{i+1}$ are all in \mathcal{L}_D for $i = 1, \dots, n-1$. Using the fact that $T^i \in \mathcal{D}$ for all $i > 0$ and that d^0 is a left \mathcal{D} -approximation of X , one can show, by Lemma 3.1, that this is equivalent to saying that $f^n - h^n d_T^{n-1} = 0, (f^0 - d_T^0 h^1) d^0 = 0$ and $f^i = h^i d_T^{i-1} + d_T^i h^{i+1}$ for all $i = 1, \dots, n-1$. Now suppose that f^\bullet is in $\text{Ker } \varphi$, and let $g : Y \oplus M \rightarrow Y \oplus M$ be in \mathcal{C} induced by that above commutative diagram, that is, $\theta(f^\bullet) = \bar{g}$. Then $\tilde{d}^n g = f^n \tilde{d}^n = h^n d_T^{n-1} \tilde{d}^n = h^n d^{n-1} \tilde{d}^n = 0$, and consequently $g \in \mathcal{R}_D$ by Lemma 3.1 (1), and $\bar{g} = 0$. Hence $\text{Ker } \varphi \subseteq \text{Ker } \theta$. Conversely, suppose that f^\bullet is a chain map in $\text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet)$ such that $\theta(f^\bullet) = \bar{g} = 0$. Then $f^n \tilde{d}^n = \tilde{d}^n g = 0$. By the fact a) above, there is a map $h^n : T^n \longrightarrow T^{n-1}$ in \mathcal{C} such that $h^n d^{n-1} = f^n$. Now $(f^{n-1} - d^{n-1} h^n) d^{n-1} = f^{n-1} d^{n-1} - d^{n-1} f^n = 0$. If $n > 1$, then by the fact a) above again, we get a morphism $h^{n-1} : T^{n-1} \longrightarrow T^{n-2}$ such that $f^{n-1} - d^{n-1} h^n = h^{n-1} d^{n-2}$. By using a) repeatedly, we get morphisms $h^i : T^i \longrightarrow T^{i-1}$ for $i = 1, \dots, n$ such that $f^n = h^n d^{n-1}$ and $f^i = h^i d^{i-1} + d^i h^{i+1}$ for all $i = 1, \dots, n$. Finally, $(f^0 - d^0 h^1) d^0 = f^0 d^0 - d^0 h^1 d^0 = d^0 f^1 - d^0 (f^1 - d^1 h^2) = 0$. Note that $d_T^i = d^i$ for $i = 0, 1, \dots, n-1$. Hence f^\bullet is in $\text{Ker } \varphi$, and consequently $\text{Ker } \theta \subseteq \text{Ker } \varphi$.

Altogether, we have shown that θ and φ are surjective ring homomorphisms with the same kernel. Hence $\text{End}_{\mathcal{C}/\mathcal{R}_D}(Y \oplus M)$ and $\text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{L}_D)}(T^\bullet)$ are isomorphism, and the theorem is proved. \square

Let A be a finite dimensional algebra, and let P be a projective A -module with $v_A P \simeq P$, where v_A is the Nakayama functor $D\text{Hom}_A(-, A)$. Suppose that Y is an A -module admitting a $\text{add}(P)$ presentation, that is, there is an exact sequence $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Y \rightarrow 0$ in $A\text{-mod}$ with $P_i \in \text{add}(P)$ for $i = 0, 1$. Let $P_2 \rightarrow \text{Ker } f_1$ be a right $\text{add}(P)$ -approximation of $\text{Ker } f_1$, we get a sequence $P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Y$. Continuing this process by taking a right $\text{add}(P)$ -approximation $P_i \rightarrow \text{Ker } f_{i-1}$ for $2 \leq i \leq n$, we get a complex

$$0 \longrightarrow X \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Y \longrightarrow 0,$$

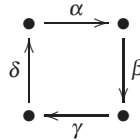
where X is the kernel of f_n .

Corollary 4.1. *Keeping the notations above, the algebras $\text{End}_A(P \oplus X)$ and $\text{End}_A(P \oplus Y)$ are derived equivalent.*

Proof. We denote the above complex by Q^\bullet with $Q^0 = X$, that is, X is in degree zero. By the construction of Q^\bullet , we have $\text{Hom}_{\mathcal{K}(A)}(P, Q^\bullet[i]) = 0$ for all i , or equivalently, $H^i(\text{Hom}_A^\bullet(P, Q^\bullet)) = 0$ for all i . Note that there is a canonical isomorphism $D\text{Hom}_A^\bullet(P, Q^\bullet) \simeq \text{Hom}_A^\bullet(Q^\bullet, v_A P)$ of complexes. Since $P \simeq v_A P$, we conclude that $H^i(\text{Hom}_A^\bullet(Q^\bullet, P)) = 0$ for all integers i . Since $\text{Hom}_A(X, -)$ is left exact, it follows clearly that $0 \longrightarrow \text{Hom}_A(X, X) \longrightarrow \text{Hom}_A(X, P_n) \longrightarrow \text{Hom}_A(X, P_{n-1})$ is exact, and consequently $H^1(\text{Hom}_A^\bullet(X, Q^\bullet)) = 0$. Similarly, we have $H^1(\text{Hom}_A^\bullet(Q^\bullet, Y)) = 0$. Set $\mathcal{D} = \text{add}(P)$. By Theorem 1.1, the algebras $\text{End}_A(P \oplus Y)/\mathcal{R}_{\mathcal{D}}(P \oplus Y)$ and $\text{End}_A(P \oplus X)/\mathcal{L}_{\mathcal{D}}(P \oplus X)$ are derived equivalent. Clearly f_0 is a surjective right \mathcal{D} -approximation. We deduce from Lemma 3.1 that $\mathcal{R}_{\mathcal{D}}(Y, P \oplus Y) = 0$. Lemma 3.1 also tells us that $\mathcal{R}_{\mathcal{D}}(P, P \oplus Y) = 0$. Hence $\mathcal{R}_{\mathcal{D}}(P \oplus Y) = 0$. Similarly one gets $\mathcal{L}_{\mathcal{D}}(P \oplus Y) = 0$, and the corollary follows. \square

Corollary 4.1 provides a convenient construction of derived equivalences. We can illustrate this with the following easy example.

Example. Let A be the Nakayama algebra give by the quiver



with relations $\alpha\beta\gamma\delta\alpha = \beta\gamma\delta\alpha\beta = \gamma\delta\alpha\beta\gamma = \delta\alpha\beta\gamma\delta = 0$. We denote by P_i the indecomposable projective A -module corresponding to the vertex i . Let $P = P_1 \oplus P_3$, and let Y be the module $\frac{1}{2}$, which admits an $\text{add}(P)$ -presentation $P_3 \rightarrow P_1 \rightarrow Y \rightarrow 0$. Using the method in Corollary 4.1, we can construct a sequence

$$0 \longrightarrow X \longrightarrow P_3 \xrightarrow{f_2} P_3 \xrightarrow{f_1} P_1 \xrightarrow{f_0} Y \longrightarrow 0,$$

where X is the module $\frac{4}{3}$. Note that the above sequence is not exact at the right P_3 . Using Corollary 4.1, we can deduce that $\text{End}_A(P_1 \oplus P_3 \oplus Y)$ and $\text{End}_A(P_1 \oplus P_3 \oplus X)$ are derived equivalent.

5 Derived equivalences and proper \mathcal{D} -annihilators

In this section, we introduce weakly n -angulated categories, and construct derived equivalences from n -angles in weakly n -angulated categories. It turns out that this new concept allows us to generalize the main results in [HKX13] and [Che13] “comfortably”, avoiding technical calculations of the morphisms in Φ -Yoneda algebras.

The notion of n -angulated category is given in [GKO13b] as a generalization of triangulated categories (In this case $n = 3$). Typical examples of n -angulated categories include certain $(n - 2)$ -cluster tilting subcategories in a triangulated category, which appear in recent cluster-tilting theory.

Definition 5.1. Let $n \geq 3$ be an integer. A weakly n -angulated k -category is an additive k -category \mathcal{C} together with an automorphism Σ of \mathcal{C} , and a class \diamond of n -angles of the form

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

satisfying the following axioms:

- (1). For each $X \in \mathcal{C}$, the sequence $X \xrightarrow{1_X} X \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma X$ belongs to \diamond . The class \diamond is closed under taking direct sums, and is closed under isomorphisms.
- (2). $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$ is in \diamond if and only if so is $X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2$.
- (3). For each commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \\ \downarrow h_1 & & \downarrow h_2 & & & & \downarrow \Sigma h_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma Y_1 \end{array}$$

with rows in \diamond , there exist morphisms $h^i : X_i \rightarrow Y_i$ for $i = 3, \dots, n$ such that

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow h_n \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma Y_1 \end{array}$$

is commutative.

Remark. (a) The relationship between weakly n -angulated categories and n -angulated categories is like the relationship between additive categories and abelian categories. In an abelian category, pullback and pushout always exist, and every morphism has a kernel and a cokernel, while additive categories do not have these properties in general. Correspondingly, an n -angulated category has a mapping cone axiom (In case $n=3$, we have Octahedral Axiom) which is similar to pullback and pushout, and every morphism can be embedded into an n -angle. However, a weakly n -angulated category does not necessarily have these properties.

(b). Just like additive categories, the axioms of Definition 5.1 can be easily satisfied by many full subcategories of n -angulated categories. Suppose that (\mathcal{C}, \diamond) is a weakly n -angulated k -category, and that \mathcal{C}' is an additive full subcategory of \mathcal{C} such that $\Sigma(\mathcal{C}') = \mathcal{C}'$. Denote by \diamond the intersection $\diamond \cap \mathcal{C}'$. Then it is easy to see that $(\mathcal{C}', \diamond')$ is again a weakly n -angulated k -category. In particular, every additive full subcategory of an n -angulated category closed under Σ and Σ^{-1} is weakly n -angulated.

An additive k -functor H from a weakly n -angulated category (\mathcal{C}, \diamond) to $k\text{-Mod}$ is called (covariantly) *cohomological*, if whenever

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

is an n -angle in \diamond , the long sequence

$$\cdots \rightarrow H(\Sigma^i X_1) \xrightarrow{H(\Sigma^i f_1)} H(\Sigma^i X_2) \xrightarrow{H(\Sigma^i f_2)} \cdots \xrightarrow{H(\Sigma^i f_{n-1})} H(\Sigma^i X_n) \xrightarrow{H(\Sigma^i f_n)} H(\Sigma^{i+1} X_1) \rightarrow \cdots$$

is exact. A contravariant additive k -functor H from (\mathcal{C}, \diamond) to $k\text{-Mod}$ is called (contravariantly) *cohomological*, if whenever

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

is an n -angle in \diamond , the long sequence

$$\cdots \longrightarrow H(\Sigma^{i+1} X_1) \xrightarrow{H(\Sigma^i f_n)} H(\Sigma^i X_n) \xrightarrow{H(\Sigma^i f_{n-1})} \cdots \xrightarrow{H(\Sigma^i f_2)} H(\Sigma^i X_2) \xrightarrow{H(\Sigma^i f_1)} H(\Sigma^i X_1) \longrightarrow \cdots$$

is exact.

Lemma 5.2. *Let \mathcal{C} be a weakly n -angulated category, and let*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

be an n -angle in \mathcal{C} . Then we have the following:

- (1). $f_i f_{i+1} = 0$ for all $i = 1, 2, \dots, n-1$;
- (2). $\mathcal{C}(X, -)$ and $\mathcal{C}(-, X)$ are cohomological;
- (3). Suppose that $2 \leq i < n$. Each commutative diagram

$$\begin{array}{ccccccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{i-1}} & X_i & \xrightarrow{f_i} & X_{i+1} & \xrightarrow{f_{i+1}} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_i & & & & & & & & \downarrow \Sigma h_1 \\ Y_1 & \xrightarrow{f_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{i-1}} & Y_i & \xrightarrow{g_i} & Y_{i+1} & \xrightarrow{g_{i+1}} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

with rows in \diamond can be completed in \mathcal{C} to a commutative diagram

$$\begin{array}{ccccccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{i-1}} & X_i & \xrightarrow{f_i} & X_{i+1} & \xrightarrow{f_{i+1}} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_i & & \downarrow h_{i+1} & & & & \downarrow h_n & & \downarrow \Sigma h_1 \\ Y_1 & \xrightarrow{f_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{i-1}} & Y_i & \xrightarrow{g_i} & Y_{i+1} & \xrightarrow{g_{i+1}} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

Proof. (1). By the axiom (2) in Definition 5.1, it suffices to prove that $f_1 f_2 = 0$. There is a diagram

$$\begin{array}{ccccccc} X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & \cdots \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2 \\ \downarrow f_2 & & \parallel & & & & \downarrow \Sigma f_2 \\ X_3 & \xrightarrow{f_2} & X_3 & \longrightarrow & 0 & \longrightarrow & \cdots \longrightarrow 0 \longrightarrow \Sigma X_3 \end{array}$$

with rows being n -angles. By axiom (3) in Definition 5.1, there is a morphism $h : \Sigma X_2 \rightarrow 0$ such that $(-1)^n \Sigma(f_1) \Sigma(f_2) = h \circ 0 = 0$. It follows that $\Sigma(f_1 f_2) = 0$, and hence $f_1 f_2 = 0$.

(2). To show that $\mathcal{C}(X, -)$ is cohomological, by the axiom (2) again, it suffices to show that the sequence

$$\mathcal{C}(X, X_1) \xrightarrow{\mathcal{C}(X, f_1)} \mathcal{C}(X, X_2) \xrightarrow{\mathcal{C}(X, f_2)} \mathcal{C}(X, X_3)$$

induced by $\mathcal{C}(X, -)$ is exact. We have shown that $f_1 f_2 = 0$. It follows that

$$\mathcal{C}(X, f_1) \mathcal{C}(X, f_2) = \mathcal{C}(X, f_1 f_2) = 0.$$

Now suppose that $g \in \mathcal{C}(X, X_2)$ is in the kernel of $\mathcal{C}(X, f_2)$, that is, $gf_2 = 0$. Then there is a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \Sigma X & \xrightarrow{(-1)^n \Sigma 1_X} & \Sigma X \\ \downarrow g & & \downarrow & & & & & & & & \downarrow \Sigma g \\ X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 & \xrightarrow{(-1)^n \Sigma f_1} & \Sigma X_2 \end{array}$$

with rows being n -angles. Thus, there exists a morphism $h : \Sigma X \rightarrow \Sigma X_1$ such that $h\Sigma(f_1) = \Sigma(1_X)\Sigma(g)$, and hence $\Sigma^{-1}(h)f_1 = g$. This shows that g is in the image of $\mathcal{C}(X, f_1)$, and the proof is finished. Dually, the functor $\mathcal{C}(-, X)$ is also cohomological.

(3). If $i = 2$, then this is just axiom (3). Now we assume that $2 < i < n$. By the axiom (3), there exist $h'_k : X_k \rightarrow Y_k, k = 3, \dots, n$, such that

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h'_3 & & & & \downarrow h'_n & & \downarrow \Sigma h_1 \\ Y_1 & \xrightarrow{f_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

is commutative. Now we have $f_2(h_3 - h'_3) = h_2g_2 - h_2g_2 = 0$. By the statement (2) that we have just proved above, there is a morphism $u_4 : X_4 \rightarrow Y_3$ such that $h_3 - h'_3 = f_3u_4$. Let $u_3 : X_3 \rightarrow Y_2$ be the zero map. Then $h_3 - h'_3 = u_3g_2 + f_3u_4$. Moreover, $f_3(h_4 - h'_4 - u_4g_3) = (h_3 - h'_3)g_3 - f_3u_4g_3 = u_3g_2g_3 = 0$. If $i \geq 4$, then $n \geq 5$ since $n > i$, and there is a morphism $u_5 : X_5 \rightarrow Y_4$ such that $h_4 - h'_4 = f_4u_5 + u_4g_3$. Continue this way, we get morphisms $u_k : X_k \rightarrow Y_{k-1}, k = 3, 4, \dots, i+1$ such that $h_k - h'_k = u_kg_{k-1} + f_ku_{k+1}$ for all $k = 3, \dots, i$. Now defining $h_{i+1} := h'_{i+1} + u_{i+1}g_i$, and $h_k := h'_k$ for all $i+1 < k \leq n$, one can easily check that the diagram in the statement (3) is commutative. \square

Now we can give a proof of our second main result.

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1. Clearly, the sequence

$$X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_n \oplus M \xrightarrow{\tilde{g}} Y \oplus M \xrightarrow{\tilde{\eta}} \Sigma X,$$

where $\tilde{g} = \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}$ is the direct sum of g and 1_M and $\tilde{\eta} := \begin{bmatrix} \eta \\ 0 \end{bmatrix}$, is still an n -angle. We denote by T^\bullet be the complex

$$0 \longrightarrow X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_n \oplus M \longrightarrow 0$$

with X in degree zero. Thus we have $T^0 = X$, $T^i = M_i$ for all $i = 1, \dots, n$, and $d_T^0 = f$ if $n > 1$. If $n = 1$, then $d_T^0 = [f, 0] : X \rightarrow M_1 \oplus M$. Applying $\mathcal{C}(M, -)$, we get an exact sequence

$$\mathcal{C}(M, X) \longrightarrow \mathcal{C}(M, M_1) \longrightarrow \cdots \longrightarrow \mathcal{C}(M, M_n \oplus M).$$

Hence $H^i(\text{Hom}_{\mathcal{C}}^\bullet(M, T^\bullet)) = 0$ for all $i \neq 0, n$. Applying $\mathcal{C}(-, M)$, we get an exact sequence

$$\mathcal{C}(M_n \oplus M, M) \longrightarrow \cdots \longrightarrow \mathcal{C}(M_1, M) \longrightarrow \mathcal{C}(X, M) \longrightarrow 0.$$

The last map is surjective because that f is a left \mathcal{D} -approximation. Hence $H^i(\text{Hom}_{\mathcal{C}}^\bullet(T^\bullet, M)) = 0$ for all $i \neq -n$. By Lemma 3.2, the complex T^\bullet is self-orthogonal in $\mathcal{K}^b(\mathcal{C}/\mathcal{I}_{\mathcal{D}})$. Similarly as in the proof of Theorem 1.1, the complex $\text{Hom}_{\mathcal{C}/\mathcal{I}_{\mathcal{D}}}^\bullet(M \oplus X, T^\bullet)$ is a tilting complex over $\text{End}_{\mathcal{C}/\mathcal{I}_{\mathcal{D}}}(M \oplus X)$ with endomorphism algebra isomorphic to $\text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{I}_{\mathcal{D}})}(T^\bullet)$. To finish to the proof, it suffices to show that $\text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{I}_{\mathcal{D}})}(T^\bullet)$ and $\text{End}_{\mathcal{C}/\mathcal{I}_{\mathcal{D}}}(Y \oplus M)$ are isomorphic.

Firstly, for each chain map f^\bullet in $\text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet)$, by Lemma 5.2 (3), there is a morphism $u \in \text{End}_{\mathcal{C}}(Y \oplus M)$ such that the diagram

$$\begin{array}{ccccccc}
 T^0 & \xrightarrow{d_T^0} & T^1 & \xrightarrow{d_T^1} & \dots & \xrightarrow{d_T^{n-1}} & T^n \xrightarrow{\tilde{g}} Y \oplus M \xrightarrow{\tilde{\eta}} \Sigma T^0 \\
 \downarrow f^0 & & \downarrow f^1 & & & & \downarrow f^n \\
 T^0 & \xrightarrow{d_T^0} & T^1 & \xrightarrow{d_T^1} & \dots & \xrightarrow{d_T^{n-1}} & T^n \xrightarrow{\tilde{g}} Y \oplus M \xrightarrow{\tilde{\eta}} \Sigma T^0 \\
 & & & & & & \downarrow u \\
 & & & & & & Y \oplus M \xrightarrow{\tilde{\eta}} \Sigma T^0 \\
 & & & & & & \downarrow \Sigma f^0
 \end{array} \quad (\star)$$

is commutative. If u' is another morphism in $\text{End}_{\mathcal{C}}(Y \oplus M)$ making the above diagram commutative, then $\tilde{g}(u - u') = 0 = (u - u')\tilde{\eta}$. Since \tilde{g} is a right \mathcal{D} -approximation by our assumption, we have $(u - u')$ belongs to $\mathcal{R}_{\mathcal{D}}(Y \oplus M, Y \oplus M)$ by Lemma 3.1 (1). It follows from $(u - u')\tilde{\eta} = 0$ that $u - u'$ factorizes through T^n , which is in \mathcal{D} . Hence $u - u'$ is in $\mathcal{J}_{\mathcal{D}}(Y \oplus M, Y \oplus M)$. Denote by \bar{u} the morphism in $\mathcal{C}/\mathcal{J}_{\mathcal{D}}$ corresponding to u . Thus, we get a map

$$\theta : \text{End}_{\mathcal{C}}(T^\bullet) \longrightarrow \text{End}_{\mathcal{C}/\mathcal{J}_{\mathcal{D}}}(Y \oplus M)$$

sending f^\bullet to \bar{u} , which is clearly a ring homomorphism. For each $u \in \text{End}_{\mathcal{C}}(Y \oplus M)$, since \tilde{g} is a right \mathcal{D} -approximation, there is $f^n : T^n \longrightarrow T^n$ such that $\tilde{g}u = f^n \tilde{g}$. Thus, by the axioms (2) and (3) in Definition 5.1, we get morphisms $f^i : T^i \longrightarrow T^i, i = 0, \dots, n$, making the above diagram commutative. This shows that θ is a surjective ring homomorphism.

Secondly, similar to that in the proof of Theorem 1.1, one can prove that there is a surjective ring homomorphism

$$\varphi : \text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet) \longrightarrow \text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{I}_{\mathcal{D}})}(T^\bullet),$$

which is the composite of the ring homomorphism $\text{End}_{\mathcal{C}^b(\mathcal{C})}(T^\bullet) \rightarrow \text{End}_{\mathcal{C}^b(\mathcal{C}/\mathcal{I}_{\mathcal{D}})}(T^\bullet)$ induced by the canonical functor $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{I}_{\mathcal{D}}$ and the canonical ring homomorphism from $\text{End}_{\mathcal{C}^b(\mathcal{C}/\mathcal{I}_{\mathcal{D}})}(T^\bullet)$ to $\text{End}_{\mathcal{K}^b(\mathcal{C}/\mathcal{I}_{\mathcal{D}})}(T^\bullet)$.

Similarly, we have to show that θ and φ have the same kernel. A chain map f^\bullet is in $\text{Ker } \varphi$ if and only if there exist $h^i : T^i \rightarrow T^{i-1}, i = 1, \dots, n$ in \mathcal{C} such that $f^0 - d_T^0 h^1, f^i - h^i d_T^{i-1} - d_T^i h^{i+1}, i = 1, \dots, n-1$, and $f^n - h^n d_T^{n-1}$ are all in $\mathcal{I}_{\mathcal{D}}$. Using the fact that $T^i \in \mathcal{D}$ for all $i > 0$ and that $d_T^0 = f$ is a left \mathcal{D} -approximation of X , one can see, by Lemma 3.1, that this is equivalent to saying that $f^n - h^n d_T^{n-1} = 0, f^i = h^i d_T^{i-1} + d_T^i h^{i+1}$ for $i = 1, \dots, n-1$, and $f^0 - d_T^0 h^1 \in \mathcal{I}_{\mathcal{D}}$.

Now suppose that f^\bullet is in $\text{Ker } \varphi$, and that $u \in \text{End}_{\mathcal{C}}(Y \oplus M)$ fits the commutative diagram (\star) above. Then $\theta(f^\bullet) = \bar{u}$. Since $f^\bullet \in \text{Ker } \varphi$, we have $f^n = h^n d_T^{n-1}$, and consequently $\tilde{g}u = f^n \tilde{g} = h^n d_T^{n-1} \tilde{g}$, which is zero by Lemma 5.2 (1). It follows from Lemma 3.1 (1) that $u \in \mathcal{R}_{\mathcal{D}}$. The fact $f^\bullet \in \text{Ker } \theta$ also implies that $f^0 - d_T^0 h^1 \in \mathcal{I}_{\mathcal{D}}$. In particular, the morphism $f^0 - d_T^0 h^1$ factorizes through an object in $\mathcal{D} = \text{add}(M)$. Assume that $f^0 - d_T^0 h^1 = ab$ for some $a \in \mathcal{C}(T^0, M')$ and $b \in \mathcal{C}(M', T^0)$ with $M' \in \mathcal{D}$. Since d_T^0 is a left \mathcal{D} -approximation, we see that a factorizes through d_T^0 , and hence $f^0 - d_T^0 h^1$ factorizes through d_T^0 . Consequently, the morphism f^0 also factorizes through d_T^0 , say, $f^0 = d_T^0 \alpha$. Thus $\tilde{\eta} \Sigma f^0 = \tilde{\eta} \Sigma(d_T^0) \Sigma(\alpha)$, which must be zero by the axiom (2) in Definition 5.1 and Lemma 5.2. Hence $u \tilde{\eta} = 0$, and consequently u factorizes through $T^n \in \mathcal{D}$ by Lemma 5.2 (2). Altogether, we have shown that u belongs to $\mathcal{R}_{\mathcal{D}} \cap \mathcal{F}_{\mathcal{D}} = \mathcal{J}_{\mathcal{D}}$. It follows that $\bar{u} = 0$ and $f^\bullet \in \text{Ker } \theta$. Hence $\text{Ker } \varphi \subseteq \text{Ker } \theta$.

Conversely, suppose that $f^\bullet \in \text{Ker } \theta$ and $u \in \text{End}_{\mathcal{C}}(Y \oplus M)$ fits the commutative diagram (\star) . Then $\theta(f^\bullet) = \bar{u} = 0$, that is, $u \in \mathcal{J}_{\mathcal{D}} = \mathcal{R}_{\mathcal{D}} \cap \mathcal{F}_{\mathcal{D}}$. Since $u \in \mathcal{R}_{\mathcal{D}}$ and \tilde{g} is a right \mathcal{D} -approximation, by Lemma 3.1 (1), we have $\tilde{g}u = 0$. Thus $f^n \tilde{g} = 0$. By Lemma 5.2 (2), there is a morphism $h^n : T^n \rightarrow T^{n-1}$ such that $f^n = h^n d_T^{n-1}$. Now $(f^{n-1} - d_T^{n-1} h^n) d_T^{n-1} = f^{n-1} d_T^{n-1} - d_T^{n-1} f^n = 0$. If $n \geq 2$, then, by Lemma 5.2 (2), there is a morphism $h^{n-1} : T^{n-1} \rightarrow T^{n-2}$ such that $f^{n-1} - d_T^{n-1} h^n = h^{n-1} d_T^{n-2}$. Moreover, $(f^{n-2} - d_T^{n-2} h^{n-1}) d_T^{n-2} = d_T^{n-2} f^{n-1} - d_T^{n-2} h^{n-1} d_T^{n-2} = d_T^{n-2} d_T^{n-1} h^n = 0$. Repeating this process, we

get $h^i : T^i \rightarrow T^{i-1}, i = 1, \dots, n$ such that $f^n = h^n d_T^{n-2}, f^i = h^i d_T^{i-1} d_T^i h^{i+1}$ for $i = 1, \dots, n-1$, and $(f^0 - d_T^0 h^1) d_T^0 = 0$. Since d_T^0 is a left \mathcal{D} -approximation, we deduce from Lemma 3.1 (2) that $f^0 - d_T^0 h^1 \in \mathcal{L}_{\mathcal{D}}$. Since $u \in \mathcal{F}_{\mathcal{D}}$ and \tilde{g} is a right \mathcal{D} -approximation, it is easy to see that u factorizes through \tilde{g} , and thus $\tilde{\eta} \Sigma(f^0) = u \tilde{\eta} = 0$. By Lemma 5.2 (2) and axiom (2) in Definition 5.1, the morphism $\Sigma(f^0)$ factorizes through $\Sigma(d_T^0)$, or equivalently, f^0 factorizes through d_T^0 . Hence $f^0 \in \mathcal{F}_{\mathcal{D}}$ and $f^0 - d_T^0 h^1 \in \mathcal{F}_{\mathcal{D}}$. Thus we have shown that $T^\bullet \in \text{Ker } \varphi$, and consequently $\text{Ker } \theta \subseteq \text{Ker } \varphi$.

Thus θ and φ have the same kernel, and the rings $\text{End}_{\mathcal{C}/\mathcal{I}_{\mathcal{D}}}(T^\bullet)$ and $\text{End}_{\mathcal{C}/\mathcal{J}_{\mathcal{D}}}(Y \oplus M)$ are isomorphic, and the theorem then follows. \square

Let (\mathcal{T}, \diamond) and $(\mathcal{T}', \diamond')$ be weakly n -angulated k -categories. An additive k -functor F from \mathcal{T} to \mathcal{T}' is called an n -angulated functor if there is a natural isomorphism $\psi : \Sigma' F \rightarrow F \Sigma$ and

$$F(X_1) \xrightarrow{F(f_1)} F(X_2) \xrightarrow{F(f_2)} \dots \xrightarrow{F(f_{n-1})} F(X_n) \xrightarrow{F(f_n) \psi_{X_1}^{-1}} \Sigma' F(X_1)$$

is in \diamond' whenever

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

is in \diamond .

Now Let (\mathcal{T}, \diamond) be a weakly n -angulated k -category, and let F be an n -angulated functor from \mathcal{T} to itself. Suppose that Φ is an admissible subset of \mathbb{Z} , and $\mathcal{T}^{F, \Phi}$ is the Φ -orbit category of \mathcal{T} . We fix a natural isomorphism $\psi(1) : \Sigma F \rightarrow F \Sigma$, and set $\psi(0) := id_\Sigma : \Sigma \rightarrow \Sigma$. For each positive integer u , we define $\psi(u) : \Sigma F^u \rightarrow F^u \Sigma$ to be the composite

$$\Sigma F^u \xrightarrow{\psi(1)_{F^{u-1}}} F \Sigma F^{u-1} \xrightarrow{F \psi(1)_{F^{u-2}}} F^2 \Sigma F^{u-2} \rightarrow \dots \xrightarrow{F^{u-1} \psi(1)} F^u \Sigma.$$

If F is not an equivalence, then $F^{-1} = 0$, and we define $\psi(u) : \Sigma F^u \rightarrow F^u \Sigma$ to be zero for all negative integers. If F is an equivalence, and F^{-1} is a quasi-inverse of F , then (F^{-1}, F) is an adjoint pair. Let $\varepsilon : id_{\mathcal{T}} \rightarrow F^{-1} F$ be the unit and let $\eta : F F^{-1} \rightarrow id_{\mathcal{T}}$ be the counit. We define $\psi(-1)$ to be the composite

$$\Sigma F^{-1} \xrightarrow{\varepsilon_{\Sigma F^{-1}}} F^{-1} F \Sigma F^{-1} \xrightarrow{F^{-1}(\psi(1)^{-1})_{F^{-1}}} F^{-1} \Sigma F F^{-1} \xrightarrow{F^{-1} \Sigma(\eta)} F^{-1} \Sigma,$$

and define $\psi(u)$, for each integer $u < 0$, to be the composite

$$\Sigma F^u \xrightarrow{\psi(-1)_{F^{u+1}}} F^{-1} \Sigma F^{u+1} \xrightarrow{F^{-1} \psi(-1)_{F^{u+2}}} F^{-2} \Sigma F^{u+2} \rightarrow \dots \rightarrow F^u \Sigma.$$

With these natural transformations in hand, we can define an automorphism Σ^Φ of $\mathcal{T}^{F, \Phi}$ as follows. $\Sigma^\Phi(X)$ is just $\Sigma(X)$ for each object X . For each homogeneous morphism $f_u : X \rightarrow F^u Y$ in $\mathcal{T}^{F, \Phi}$, we define $\Sigma^\Phi(f_u)$ to be the composite

$$\Sigma X \xrightarrow{\Sigma(f_u)} \Sigma F^u Y \xrightarrow{\psi(u)_Y} F^u \Sigma Y.$$

One can check that Σ^Φ is indeed an automorphism of the Φ -orbit category $\mathcal{T}^{F, \Phi}$. Let \diamond^Φ be the sequences in $\mathcal{T}^{F, \Phi}$ isomorphic to those n -angles in \diamond .

Proposition 5.3. *Keeping the notations above, the Φ -orbit category $\mathcal{T}^{F, \Phi}$, together with Σ^Φ and \diamond^Φ , is a weakly n -angulated category.*

Proof. The axioms (1) and (2) of Definition 5.1 are satisfied by the definition of \diamond^Φ and Σ^Φ . Now given a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma^\Phi X_1 \\ \downarrow h^1 & & \downarrow h^2 & & & & \downarrow \Sigma^\Phi(h^1) \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma^\Phi Y_1 \end{array}$$

in $\mathcal{T}^{F,\Phi}$ with rows in \diamond^Φ . Clearly, we can assume that the rows are in \diamond , and all the morphisms f_i are homogeneous morphisms of degree zero for all $i = 1, \dots, n$. Let $h^1 = (h_u^1)_{u \in \Phi}$ and $h^2 = (h_u^2)_{u \in \Phi}$. Then $h_u^1 g_1 = f_1 h_u^2$ for all $u \in \Phi$. Thus, we get a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \\ \downarrow h_u^1 & & \downarrow h_u^2 & & & & \downarrow \Sigma(h_u^1) \\ F^u(Y_1) & \xrightarrow{F^u(g_1)} & F^u(Y_2) & \xrightarrow{F^u(g_2)} & F^u(Y_3) & \xrightarrow{F^u(g_3)} & \cdots \xrightarrow{F^u(g_{n-1})} F^u(Y_n) \xrightarrow{F^u(g_n)\psi(u)^{-1}} \Sigma F^u(Y_1) \end{array}$$

in \mathcal{T} with rows in \diamond . Thus, in the weakly n -angulated category \mathcal{T} , we get a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \\ \downarrow h_u^1 & & \downarrow h_u^2 & & \downarrow h_u^3 & & \downarrow h_u^n \\ F^u(Y_1) & \xrightarrow{F^u(g_1)} & F^u(Y_2) & \xrightarrow{F^u(g_2)} & F^u(Y_3) & \xrightarrow{F^u(g_3)} & \cdots \xrightarrow{F^u(g_{n-1})} F^u(Y_n) \xrightarrow{F^u(g_n)\psi(u)^{-1}} \Sigma F^u(Y_1) \end{array}$$

for all $u \in \Phi$. Defining $h^i := (h_u^i)_{u \in \Phi}$, we obtain a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma^\Phi X_1 \\ \downarrow h^1 & & \downarrow h^2 & & \downarrow h^3 & & \downarrow h^n \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma^\Phi Y_1 \end{array}$$

in $\mathcal{T}^{F,\Phi}$. Thus $\mathcal{T}^{F,\Phi}$ satisfies the axiom (3) of Definition 5.1. Hence $\mathcal{T}^{F,\Phi}$, together with Σ^Φ and \diamond^Φ is a weakly n -angulated category. \square

Let us remark that, when \mathcal{T} is an n -angulated category, the Φ -orbit category $\mathcal{T}^{F,\Phi}$ is not an n -angulated category in general.

Let \mathcal{T} be a weakly n -angulated k -category with an n -angulated endo-functor F . Suppose that Φ is an admissible subset of \mathbb{Z} . Let M be an object in \mathcal{T} and set $\mathcal{D} := \text{add}_{\mathcal{T}^{F,\Phi}}(M)$. Suppose that

$$X \xrightarrow{f} M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_{n-2} \xrightarrow{g} Y \xrightarrow{w} \Sigma X$$

is an n -angle in \mathcal{T} with all $M_i \in \text{add}(M)$. Set $\bar{w} := \begin{bmatrix} g \\ 0 \end{bmatrix} : Y \oplus M \rightarrow \Sigma X$ and $\tilde{w} := [w, 0] : Y \rightarrow \Sigma X \oplus M$, and define

$$I := \left\{ (x_i) \in E_{\mathcal{T}}^{F,\Phi}(X \oplus M) \mid x_i = 0, \forall 0 \neq i \in \Phi, x_0 \text{ factorizes through } \text{add}(M) \text{ and } \Sigma^{-1}(\tilde{w}) \right\}$$

$$J := \left\{ (x_i) \in E_{\mathcal{T}}^{F,\Phi}(Y \oplus M) \mid x_i = 0, \forall 0 \neq i \in \Phi, x_0 \text{ factorizes through } \text{add}(M) \text{ and } \bar{w} \right\}.$$

This was defined in [HKX13], where \mathcal{T} is a triangulated category and F is a triangulated auto-equivalence. There it is proved that I (respectively, J) is an ideal provided that f is a left (respectively, right) \mathcal{D} -approximation in the Φ -orbit category $\mathcal{T}^{F,\Phi}$ and $\mathcal{T}(F^i M, X) = 0$ (respectively, $\mathcal{T}(Y, F^i M) = 0$) for all $0 \neq i \in \Phi$. The following lemma shows that the ideals I and J are actually proper \mathcal{D} -annihilators.

Lemma 5.4. *Keeping the notations above, we have the following.*

(1). *If g is a right \mathcal{D} -approximation in $\mathcal{T}^{F,\Phi}$, and $\mathcal{T}(Y, F^i M) = 0$ for all $0 \neq i \in \Phi$, then $J = \mathcal{J}_{\mathcal{D}}(Y \oplus M)$;*

(2). *If f is a left \mathcal{D} -approximation in $\mathcal{T}^{F,\Phi}$, and $\mathcal{T}(M, F^i X) = 0$ for all $0 \neq i \in \Phi$, then $I = \mathcal{I}_{\mathcal{D}}(X \oplus M)$.*

Proof. (1). By Lemma 3.1, we have $\mathcal{J}_{\mathcal{D}}(M, Y \oplus M) = 0$. Now we consider $\mathcal{J}_{\mathcal{D}}(Y, Y \oplus M)$. Since g is a right \mathcal{D} -approximation. By Lemma 3.1 (1), one has $\mathcal{J}_{\mathcal{D}}(Y, Y \oplus M)$ consists of those morphisms $(x_i) \in E_{\mathcal{T}}^{F,\Phi}(Y, Y \oplus M)$ satisfying the conditions:

(a). $gx_i = 0$ for all $i \in \Phi$;

(b). There is some $(y_i) \in \mathcal{T}^{F,\Phi}(Y, M_Y \oplus M)$ such that $y_i * \tilde{g} = x_i$ for all $i \in \Phi$.

By our assumption that $\mathcal{T}(Y, F^i M) = 0$ for all $0 \neq i \in \Phi$, the morphism y_i in condition (b) above is zero for all $0 \neq i \in \Phi$. Thus $x_i = 0$ for all $0 \neq i \in \Phi$, $gx_0 = 0$ and x_0 factorizes through $\text{add}(M)$ in \mathcal{T} . The proof of (2) is dual. \square

Combining Theorem 1.2 and Lemma 5.4, we get the following corollary.

Corollary 5.5. *Let (\mathcal{T}, Σ) be a weakly n -angulated category ($n \geq 3$) with an n -angulated endo-functor, and let M be an object in \mathcal{T} . Suppose that Φ is an admissible subset of \mathbb{Z} . Let*

$$X \xrightarrow{f} M_1 \longrightarrow \cdots \longrightarrow M_{n-2} \xrightarrow{g} Y \longrightarrow \Sigma X$$

be an n -angle in \mathcal{T} such that $M_i \in \text{add}(M)$ for all $i = 1, \dots, n-2$, and that f and g are left and right $\text{add}(M)$ -approximations in the orbit category $\mathcal{T}^{F,\Phi}$, respectively. Suppose that $\mathcal{T}(Y, F^i M) = 0 = \mathcal{T}(M, F^i X)$ for all $0 \neq i \in \Phi$. Then the rings $E_{\mathcal{T}}^{F,\Phi}(M \oplus X)/I$ and $E_{\mathcal{T}}^{F,\Phi}(M \oplus Y)/J$ are derived equivalent.

This corollary generalizes the results [HKX13, Theorem 3.1] and [Che13, Theorem 1.1]: the functor F here is not necessarily an auto-equivalence, while this is required in both [HKX13] and [Che13].

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